

Relation between the skew-rank of an oriented graph and the independence number of its underlying graph*

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Abstract

An oriented graph G^σ is a digraph without loops or multiple arcs whose underlying graph is G . Let $S(G^\sigma)$ be the skew-adjacency matrix of G^σ and $\alpha(G)$ be the independence number of G . The rank of $S(G^\sigma)$ is called the skew-rank of G^σ , denoted by $sr(G^\sigma)$. Wong et al. [European J. Combin. 54 (2016) 76-86] studied the relationship between the skew-rank of an oriented graph and the rank of its underlying graph. In this paper, the correlation involving the skew-rank, the independence number, and some other parameters are considered. First we show that $sr(G^\sigma) + 2\alpha(G) \geq 2|V_G| - 2d(G)$, where $|V_G|$ is the order of G and $d(G)$ is the dimension of cycle space of G . We also obtain sharp lower bounds for $sr(G^\sigma) + \alpha(G)$, $sr(G^\sigma) - \alpha(G)$, $sr(G^\sigma)/\alpha(G)$ and characterize all corresponding extremal graphs.

Keywords: Skew-rank; Oriented graph; Evenly-oriented; Independence number

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1 Introduction

We will start with introducing some background information that will lead to our main results. Some important previously established facts will also be presented.

1.1 Background

Let $G = (V_G, E_G)$ be a graph with vertex set $V_G = \{v_1, v_2, \dots, v_n\}$ and edge set E_G . Denote by P_n, C_n and K_n a path, a cycle and a complete graph of order n , respectively. The set of

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neighbors of a vertex v in G is denoted by $N_G(v)$ or simply $N(v)$. Unless otherwise stated, we follow the traditional notations and terminologies (see, for instance, [10]).

The *adjacency matrix* $A(G)$ of G is an $n \times n$ matrix whose (i, j) -entry is 1 if vertices v_i and v_j are adjacent and 0 otherwise. Given a graph G , the oriented graph G^σ is obtained from G by assigning each edge of G a direction. We call G the *underlying graph* of G^σ . The *skew-adjacency matrix* associated to G^σ , denoted by $S(G^\sigma)$, is defined to be an $n \times n$ matrix $[s_{x,y}]$ such that $s_{x,y} = 1$ if there is an arc from x to y , $s_{x,y} = -1$ if there is an arc from y to x and $s_{x,y} = 0$ otherwise. The *rank* of G , denoted by $r(G)$, is the rank of $A(G)$. The *skew-rank* of G^σ , denoted by $sr(G^\sigma)$, is the rank of $S(G^\sigma)$. It is easy to see that $sr(G^\sigma)$ is even since $S(G^\sigma)$ is skew symmetric.

The value

$$d(G) := |E_G| - |V_G| + \omega(G),$$

is called the *dimension* of cycle space of G , where $\omega(G)$ is the number of the components of G . Two distinct edges in a graph G are *independent* if they do not share a common end-vertex. A *matching* is a set of pairwise independent edges of G , while a *maximum matching* of G is a matching with the maximum cardinality. The *matching number* of G , written as $\alpha'(G)$, is the cardinality of a maximum matching of G . Two vertices of a graph G are said to be *independent* if they are not adjacent. A subset I of V_G is called an *independent set* if any two vertices of I are independent in G . An independent set I is *maximum* if G has no independent set I' with $|I'| > |I|$. The number of vertices in a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha(G)$. An oriented graph is called *acyclic* (resp. *connected*, *bipartite*) if its underlying graph is acyclic (resp. connected, bipartite). A graph is called an *empty graph* if it has no edges. We call v a *cut-vertex* of a connected G^σ if $G^\sigma - v$ is disconnected.

The study on skew spectrum of oriented graphs has attracted much attention. Anuradha and Balakrishnan [2] investigated skew spectrum of the Cartesian product of two oriented graphs. Anuradha et al. [3] considered the skew spectrum of bipartite graphs. Hou and Lei [12] studied the coefficients of the characteristic polynomial of skew-adjacency matrix of a oriented graph. Xu [27] established a relation between the spectral radius and the skew spectral radius. Cavers et al. [9] systematically studied skew-adjacency matrices of directed graphs. Among various specific topics, the minimal skew-rank of oriented graphs is of particular interest to researchers. The graphs with minimum skew rank 2, 4 are characterized in [1]. The bicyclic oriented graphs with skew-rank 2, 4 and 6 are, respectively, determined in [22] and [15]. Recently, Mallik and Shader [19] studied the minimum rank of all real skew-symmetric matrices described by a graph. Wong, Ma and Tian [26] presented a beautiful relation between the skew-rank of an oriented graph and the rank of its underlying graph. Huang and Li [13] further extended these results. For more properties and applications of the skew-rank of oriented graphs, we refer the readers to [18, 20, 21, 23].

Very recently, Ma, Wong and Tian [16] determined the relationship between $sr(G^\sigma)$ and the matching number $\alpha'(G)$. They [17] also characterized the relationship between $r(G)$ and $p(G)$ (the number of pendant vertices of G), from which one can obtain the same relationship between $sr(G^\sigma)$ and $p(G)$. It is natural to further this study by considering the correlation between $sr(G^\sigma)$ and some other parameters of its underlying graph. In this paper we first establish the sharp lower bound on $sr(G^\sigma) + 2\alpha(G)$ of an oriented graph. We then apply the same fundamental idea to determine sharp lower bounds on $sr(G^\sigma) + \alpha(G)$, $sr(G^\sigma) - \alpha(G)$ and $sr(G^\sigma)/\alpha(G)$ and characterize the corresponding extremal oriented graphs.

1.2 Main results

Let $C_k = v_1 v_2 \cdots v_k v_1$ be a cycle of length k . The *sign* of C_k^σ with respect to σ is defined to be the sign of $\left(\prod_{i=1}^{k-1} s_{i,i+1}\right) \cdot s_{k,1}$. An even oriented cycle C_k^σ is called *evenly-oriented* (resp. *oddly-oriented*) if its sign is positive (resp. negative). An induced subgraph of G^σ is an induced subgraph of G where each edge preserves the original orientation in G^σ . For an induced subgraph H^σ of G^σ , let $G^\sigma - H^\sigma$ be the subgraph obtained from G^σ by removing all vertices of H^σ and their incident edges. For $W \subseteq V_{G^\sigma}$, $G^\sigma - W$ is the subgraph obtained from G^σ by removing all vertices in W and all incident edges. A vertex of G^σ is called a *pendant vertex* if it is of degree one in G , whereas a vertex of G^σ is called a *quasi-pendant vertex* if it is adjacent to a pendant vertex in G .

Given a graph G with pairwise vertex-disjoint cycles, let \mathcal{C}_G denote the set of all cycles of G . Contracting each cycle to a single vertex yields an acyclic graph T_G from G . It is clear that T_G is always acyclic. Note that the graph $T_G - W_\mathcal{C}$ (where $W_\mathcal{C}$ is the set of vertices corresponding to the cycles in G) is the same as the graph obtained from G by removing all the vertices on cycles and their incident edges. We denote this graph by Γ_G . For example, in Fig. 1, T_G is obtained from G by contracting each cycle into a single vertex, and Γ_G is obtained from G by removing all the vertices on cycles and their incident edges.

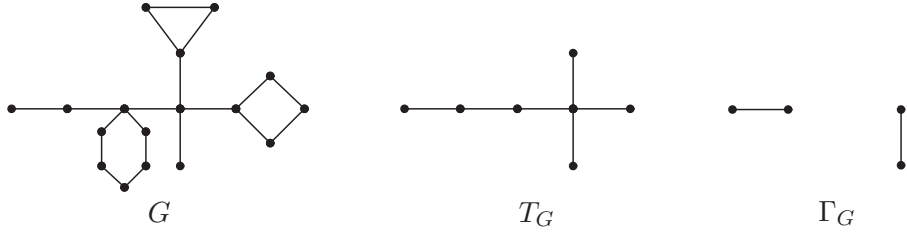


Figure 1: Graphs G , T_G , and Γ_G .

Following the above notations our first main result reads as follows.

Theorem 1.1. *Let G^σ be a simple connected graph on n vertices. Then*

$$sr(G^\sigma) + 2\alpha(G) \geq 2n - 2d(G). \quad (1.1)$$

The equality in (1.1) holds if and only if the following conditions hold for G^σ :

- (i) *the cycles (if any) of G^σ are pairwise vertex-disjoint;*
- (ii) *each cycle of G^σ is odd or evenly-oriented;*
- (iii) $\alpha(T_G) = \alpha(\Gamma_G) + d(G)$.

For example, let G be as in Fig. 1. If all the even cycles in G^σ are evenly-oriented, then G^σ satisfies conditions (i)-(iii) (note that $\alpha(T_G) = 5$, $\alpha(\Gamma_G) = 2$, $d(G) = 3$) and $sr(G^\sigma) + 2\alpha(G) = 2n - 2d(G)$ holds with $n = 17$, $sr(G^\sigma) = 12$ and $\alpha(G) = 8$.

In the case that G is bipartite, the following is a direct consequence of Theorem 1.1.

Corollary 1.2. *Let G^σ be a simple connected bipartite graph with n vertices. Then $sr(G^\sigma) + 2\alpha(G) \geq 2n - 2d(G)$ with equality if and only if the following conditions hold for G^σ :*

- (i) the cycles (if any) of G^σ are pairwise vertex-disjoint;
- (ii) each cycle of G^σ is evenly-oriented;
- (iii) $\alpha(T_G) = \alpha(\Gamma_G) + d(G)$.

Note that $\alpha(G) + \alpha'(G) = |V_G|$ if G is bipartite and $d(G)$ is exactly the number of cycles if the cycles of G are pairwise vertex-disjoint. Then Corollary 1.2 is equivalent to Theorem 1.3 below when G is bipartite, obtained in [16], showing the correlation between the skew-rank of an oriented graph, the matching number, and the dimension of cycle space of its underlying graph.

Theorem 1.3 ([16]). *Let G^σ be a simple connected graph. Then $sr(G^\sigma) - 2\alpha'(G) \geq -2d(G)$ with equality if and only if the following conditions hold for G^σ :*

- (i) the cycles (if any) of G^σ are pairwise vertex-disjoint;
- (ii) each cycle of G^σ is evenly-oriented;
- (iii) $\alpha'(T_G) = \alpha'(\Gamma_G)$.

Along the same line, we establish sharp lower bounds on $sr(G^\sigma) + \alpha(G)$, $sr(G^\sigma) - \alpha(G)$, and $sr(G^\sigma)/\alpha(G)$ in the next three theorems.

Theorem 1.4. *Let G^σ be a simple connected graph with n vertices and m edges. Then*

$$sr(G^\sigma) + \alpha(G) \geq 4n - 2m - \sqrt{n(n-1) - 2m} + \frac{1}{4} - \frac{5}{2} \quad (1.2)$$

with equality if and only if $G \cong S_n$ or $G \cong C_3$.

Theorem 1.5. *Let G^σ be a simple connected graph with n vertices and m edges. Then*

$$sr(G^\sigma) - \alpha(G) \geq 4n - 2m - 3\sqrt{n(n-1) - 2m} + \frac{1}{4} - \frac{7}{2}$$

with equality if and only if $G \cong S_n$ or $G \cong C_3$.

Theorem 1.6. *Let G^σ be a simple connected graph with n vertices and m edges. Then*

$$\frac{sr(G^\sigma)}{\alpha(G)} \geq \frac{4(2n - m - 1)}{\sqrt{4n(n-1) - 8m + 1} + 1} - 2$$

with equality if and only if $G \cong S_n$ or $G \cong C_3$.

In the rest of this section we recall some important known results. In Section 2 we first establish some technical lemmas that help us characterize the extremal graphs. We present the proofs of our main results in Section 3. We briefly comment on our findings and propose some questions in Section 4.

1.3 Preliminaries

For the rest of our introduction we recall the following important facts.

Lemma 1.7 ([14]). *Let G^σ be an oriented graph:*

- (i) If H^σ is an induced subgraph of G^σ , then $sr(H^\sigma) \leq sr(G^\sigma)$;

- (ii) If $G_1^\sigma, G_2^\sigma, \dots, G_t^\sigma$ are all the components of G^σ , then $sr(G^\sigma) = \sum_{i=1}^t sr(G_i^\sigma)$;
- (iii) $sr(G^\sigma) \geq 0$ with equality if and only if G^σ is an empty graph.

The following observation immediately follows from the definition of the independence number.

Lemma 1.8. *Let G be a simple connected graph. Then*

- (i) $\alpha(G) - 1 \leq \alpha(G - v) \leq \alpha(G)$ for any $v \in V_G$;
- (ii) $\alpha(G - e) \geq \alpha(G)$ for any $e \in E_G$.

Lemma 1.9 ([10]). *Let P_n be a path of order n . Then $r(P_n) = n$ if n is even, and $r(P_n) = n - 1$ if n is odd.*

Lemma 1.10 ([14]). *Let F^σ be an oriented acyclic graph with matching number $\alpha'(F)$. Then $sr(F^\sigma) = r(F) = 2\alpha'(F)$.*

Lemma 1.11 ([6]). *Let G be a bipartite graph with n vertices. Then $\alpha(G) + \alpha'(G) = n$.*

Lemma 1.12 ([24]). *Let C_n^σ be an oriented cycle of order n . Then $sr(C_n^\sigma) = n$ if C_n^σ is oddly-oriented, $sr(C_n^\sigma) = n - 2$ if C_n^σ is evenly-oriented and $sr(C_n^\sigma) = n - 1$ if n is odd.*

Lemma 1.13 ([14]). *Let y be a pendant vertex of G^σ , and x be the neighbor of y , then $sr(G^\sigma) = sr(G^\sigma - x) + 2 = sr(G^\sigma - x - y) + 2$.*

Lemma 1.14 ([26]). *Let x be a vertex of G^σ . Then $sr(G^\sigma - x)$ is equal to either $sr(G^\sigma)$ or $sr(G^\sigma) - 2$.*

The following lemma on the dimension of cycle space of G follows directly from the definition of $d(G)$.

Lemma 1.15 ([26]). *Let G be a graph with $x \in V_G$.*

- (i) $d(G) = d(G - x)$ if x is not on any cycle of G ;
- (ii) $d(G - x) \leq d(G) - 1$ if x lies on a cycle;
- (iii) $d(G - x) \leq d(G) - 2$ if x is a common vertex of distinct cycles;
- (iv) If the cycles of G are pairwise vertex-disjoint, then $d(G)$ is exactly the number of cycles in G .

The next result is on the rank of an acyclic graph. Let T be a tree with at least one edge, we denote by \tilde{T} the subtree obtained from T by removing all pendant vertices of T .

Lemma 1.16 ([16]). *Let T be a tree with at least one edge. Then*

- (i) $r(\tilde{T}) < r(T)$;
- (ii) If $r(T - D) = r(T)$ for a subset D of V_T , then there is a pendant vertex v such that $v \notin D$.

Recall that $p(G)$ is the number of pendant vertices of G , from Lemmas 1.10, 1.11 and 1.16 we immediately have the following.

Corollary 1.17. *Let T be a tree with at least one edge. Then*

- (i) $\alpha(T) < \alpha(\tilde{T}) + p(T)$;
- (ii) If $\alpha(T) = \alpha(T - D) + |D|$ for a subset D of V_T , then there is a pendant vertex v such that $v \notin D$.

2 Technical lemmas

In this section we present a few technical lemmas. First we establish (1.1).

Lemma 2.1. *The inequality (1.1) holds.*

Proof. We proceed by induction on $d(G)$. If $d(G) = 0$, then G^σ is an oriented tree and the result follows immediately from Lemmas 1.10 and 1.11. Now suppose that G^σ has at least one cycle, i.e., $d(G) \geq 1$, and let x be a vertex on some cycle. By Lemma 1.15(ii) we have

$$d(G - x) \leq d(G) - 1. \quad (2.3)$$

By the induction hypothesis one has

$$sr(G^\sigma - x) + 2\alpha(G - x) \geq 2(n - 1) - 2d(G - x). \quad (2.4)$$

By Lemma 1.7(i) and Lemma 1.8(i), we obtain

$$sr(G^\sigma - x) \leq sr(G^\sigma), \quad \alpha(G - x) \leq \alpha(G). \quad (2.5)$$

The inequality (1.1) then follows from (2.3)-(2.5). \square

For convenience we call a graph G^σ “lower optimal” if it achieves equality in (1.1). In the rest of this section we aim to provide some fundamental characterizations of lower-optimal oriented graphs.

Lemma 2.2. *Let x be a vertex on a cycle of G^σ . If G^σ is lower-optimal, then*

- (i) $sr(G^\sigma) = sr(G^\sigma - x)$;
- (ii) $\alpha(G) = \alpha(G - x)$;
- (iii) $d(G) = d(G - x) + 1$;
- (iv) $G^\sigma - x$ is lower-optimal;
- (v) x lies on just one cycle of G and x is not a quasi-pendant vertex of G .

Proof. The lower-optimal condition for G^σ together with the proof of Lemma 2.1 forces equalities in (2.3)-(2.5). Consequently we have (i)-(iv). By (iii) and Lemma 1.15(iii) we obtain that x lies on just one cycle of G . If x is a quasi-pendant vertex adjacent to a pendant vertex y , then by Lemma 1.13, we have $sr(G^\sigma) = sr(G^\sigma - x) + 2$, which is a contradiction to (i). This completes the proof of (v). \square

The next observation, although simple, is very helpful to our proof.

Lemma 2.3. *Let y be a pendant vertex of G with neighbor x . Then $\alpha(G) = \alpha(G - x) = \alpha(G - x - y) + 1$.*

Proof. It is routine to check that $\alpha(G - x) = \alpha(G - x - y) + 1$. In order to complete the proof, it suffices to show that $\alpha(G) = \alpha(G - x)$. In fact, let I be a maximum independent set of G .

If $x \notin I$, then I is also a maximum independent set of $G - x$ and we have $\alpha(G) = |I| = \alpha(G - x)$.

If $x \in I$, then $y \notin I$, thus $(I \setminus \{x\}) \cup \{y\}$ is an independent set of $G - x$. Hence we have $\alpha(G - x) \geq |(I \setminus \{x\}) \cup \{y\}| = |I| = \alpha(G)$. By Lemma 1.8(i), we have $\alpha(G - x) \leq \alpha(G)$.

Therefore we have $\alpha(G) = \alpha(G - x) = \alpha(G - x - y) + 1$ as desired. \square

Given an induced oriented subgraph H^σ of G^σ , let v_i be in $V_{G^\sigma} \setminus V_{H^\sigma}$. Then the induced oriented subgraph of G^σ with vertex set $V_{H^\sigma} \cup \{v_i\}$ is simply written as $H^\sigma + v_i$. The following lemma summarizes a few known results.

Lemma 2.4. *Let C_q^σ be a pendant oriented cycle of G^σ with x being a vertex of C_q of degree 3, and let $H^\sigma = G^\sigma - C_q^\sigma$, $M^\sigma = H^\sigma + x$. Then*

$$sr(G^\sigma) = \begin{cases} q - 1 + sr(M^\sigma), & \text{if } q \text{ is odd;} & (\text{see [26]}) \\ q - 2 + sr(M^\sigma), & \text{if } C_q^\sigma \text{ is evenly-oriented;} & (\text{see [26]}) \\ q + sr(H^\sigma), & \text{if } C_q^\sigma \text{ is oddly-oriented.} & (\text{see [13]}) \end{cases}$$

Following the same direction we establish a few more facts in the rest of this section.

Lemma 2.5. *Let C_q^σ be a pendant oriented cycle of G^σ with x being the unique vertex of C_q of degree 3. Let $H^\sigma = G^\sigma - C_q^\sigma$ and $M^\sigma = H^\sigma + x$, if G^σ is lower-optimal, then*

- (i) q is odd or C_q^σ is evenly-oriented;
- (ii) $sr(G^\sigma) = q - 1 + sr(H^\sigma)$, $\alpha(G) = \alpha(H) + \frac{q-1}{2}$ if q is odd and $sr(G^\sigma) = q - 2 + sr(H^\sigma)$, $\alpha(G) = \alpha(H) + \frac{q}{2}$ if C_q^σ is evenly-oriented;
- (iii) both H^σ and M^σ are lower-optimal;
- (iv) $sr(M^\sigma) = sr(H^\sigma)$ and $\alpha(M) = \alpha(H) + 1$.

Proof. (i) Supposing for contradiction that C_q^σ is oddly-oriented, then by Lemma 2.4 we have

$$sr(G^\sigma) = q + sr(H^\sigma). \quad (2.6)$$

Note that x lies on the cycle C_q . Hence, by Lemma 2.2(ii) we have

$$\alpha(G) = \alpha(G - x) = \alpha(P_{q-1}) + \alpha(H) = \frac{q}{2} + \alpha(H). \quad (2.7)$$

As C_q is a pendant cycle of G , we have

$$d(G) = d(M) + 1 = d(H) + 1. \quad (2.8)$$

Suppose $|V_G| = n$. Since G^σ is lower-optimal, we have

$$sr(G^\sigma) + 2\alpha(G) = 2n - 2d(G). \quad (2.9)$$

From (2.6)-(2.9) we have $sr(H^\sigma) + 2\alpha(H) = 2(n - q) - 2d(H) - 2$, which is a contradiction to (1.1). This completes the proof of (i).

Next we show (ii)-(iv) according to the following two possible cases.

Case 1. q is odd.

(ii) Note that x lies on a cycle of G , by Lemma 2.2(i)-(ii) we have

$$sr(G^\sigma) = sr(G^\sigma - x) = sr(P_{q-1}^\sigma) + sr(H^\sigma) = q - 1 + sr(H^\sigma), \quad (2.10)$$

$$\alpha(G) = \alpha(G - x) = \alpha(P_{q-1}) + \alpha(H) = \frac{q-1}{2} + \alpha(H). \quad (2.11)$$

(iii)-(iv) From (2.8)-(2.11) we have $sr(H^\sigma) + 2\alpha(H) = 2(n - q) - 2d(H)$, implying that H^σ is lower-optimal.

Since q is odd, by Lemma 2.4, we have

$$sr(G^\sigma) = q - 1 + sr(M^\sigma). \quad (2.12)$$

Combining (2.10) and (2.12) yields

$$sr(H^\sigma) = sr(M^\sigma). \quad (2.13)$$

Furthermore,

$$\begin{aligned} 2\alpha(H) &= 2(n - q) - sr(H^\sigma) - 2d(H) \\ &= 2(n - q + 1) - sr(M^\sigma) - 2d(H) - 2 \\ &= 2(n - q + 1) - sr(M^\sigma) - 2d(M) - 2 \\ &\leq 2\alpha(M) - 2, \end{aligned} \quad (2.14)$$

where the first equality follows from the lower-optimal condition for H^σ , the second and the third equalities follow from (2.13) and (2.8), respectively. And the last inequality (2.14) follows from applying (1.1) to M^σ . Thus we have $\alpha(H) \leq \alpha(M) - 1$. It follows from Lemma 1.8(i) that $\alpha(H) \geq \alpha(M) - 1$. Hence $\alpha(H) = \alpha(M) - 1$.

Consequently we have $sr(M^\sigma) + 2\alpha(M) = 2(n - q + 1) - 2d(M)$, implying that M^σ is also lower-optimal.

Case 2. C_q^σ is evenly-oriented.

(ii) Since x lies on a cycle of G , by Lemma 2.2(i)-(ii) we have

$$sr(G^\sigma) = sr(G^\sigma - x) = sr(P_{q-1}^\sigma) + sr(H^\sigma) = q - 2 + sr(H^\sigma), \quad (2.15)$$

$$\alpha(G) = \alpha(G - x) = \alpha(P_{q-1}) + \alpha(H) = \frac{q}{2} + \alpha(H). \quad (2.16)$$

(iii) Let x_1 be on C_q such that it is adjacent to x . By applying Lemma 2.2 to G^σ (resp. G) and Lemma 1.13 (resp. Lemma 2.3) to $G^\sigma - x_1$ (resp. $G - x_1$) we have

$$sr(G^\sigma) = sr(G^\sigma - x_1) = q - 2 + sr(M^\sigma), \quad (2.17)$$

$$\alpha(G) = \alpha(G - x_1) = \frac{q - 2}{2} + \alpha(M). \quad (2.18)$$

From (2.8)-(2.9) and (2.15)-(2.16), one has $sr(H^\sigma) + 2\alpha(H) = 2(n - q) - 2d(H)$, implying that H^σ is lower-optimal.

Combining (2.8)-(2.9) and (2.17)-(2.18), we have $sr(M^\sigma) + 2\alpha(M) = 2(n - q + 1) - 2d(M)$, which implies that M^σ is also lower-optimal.

(iv) Combining (2.15) and (2.17) yields $sr(M^\sigma) = sr(H^\sigma)$, whereas equalities (2.16) and (2.18) lead to $\alpha(M) = \alpha(H) + 1$.

This completes the proof. \square

Lemma 2.6. *Let y be a pendant vertex of G^σ with neighbor x , and let $H^\sigma = G^\sigma - y - x$. If G^σ is lower-optimal, then*

(i) x does not lie on any cycle of G ;

(ii) H^σ is also lower-optimal.

Proof. (i) Since x is a quasi-pendant vertex of G , Lemma 2.2(v) states that x does not lie on any cycle of G .

(ii) By Lemmas 1.13 and 2.3, we have

$$sr(G^\sigma) = sr(H^\sigma) + 2 \text{ and } \alpha(G) = \alpha(H) + 1. \quad (2.19)$$

Since x does not lie on any cycle of G , by Lemma 1.15(i) we have

$$d(G) = d(H). \quad (2.20)$$

Equalities (2.19)-(2.20), together with the lower-optimal condition of G^σ , imply that $sr(H^\sigma) + 2\alpha(H) = 2(n-2) - 2d(H)$, i.e., H^σ is lower-optimal. \square

Lemma 2.7. *If G^σ is lower-optimal, then*

- (i) *the cycles (if any) of G^σ are pairwise vertex-disjoint;*
- (ii) *each cycle (if any) of G^σ is odd or evenly-oriented;*
- (iii) $\alpha(G) = \alpha(T_G) + \sum_{C \in \mathcal{C}_G} \left\lfloor \frac{|V_C|}{2} \right\rfloor - d(G).$

Proof. If G contains cycles, then let x be a vertex on some cycle. By Lemma 2.2(iii) we have $d(G) = d(G-x) + 1$. By Lemma 1.15(iii) x can not be a common vertex of distinct cycles, hence the cycles of G^σ are pairwise vertex-disjoint. This completes the proof of (i).

We proceed by induction on the order n of G to prove (ii) and (iii). The initial case $n = 1$ is trivial. Suppose that (ii) and (iii) hold for any lower-optimal oriented graph of order smaller than n , and suppose G^σ is an lower-optimal oriented graph of order $n \geq 2$.

If T_G is an empty graph, then G^σ is a single oriented cycle. Thus (ii) follows from the fact that a single oriented cycle C_q^σ is lower-optimal if and only if q is odd or C_q^σ itself is evenly-oriented. And (iii) follows from the fact that $\alpha(C_q) = \frac{q-1}{2}$ if q is odd and $\alpha(C_q) = \frac{q}{2}$ if C_q^σ is evenly-oriented.

If T_G has at least one edge, then T_G contains at least one pendant vertex, say y . Then y is either a pendant vertex of G or $y \in W_\mathcal{C}$, in which case G contains a pendant cycle. We now consider both cases.

Case 1. G contains a pendant vertex y . In this case, let x be the neighbor of y in G and let $H^\sigma = G^\sigma - x - y$. By Lemma 2.6, x is not a vertex on any cycle of G and H^σ is also lower-optimal. By induction hypothesis we have

- (a) each cycle (if any) of H^σ is odd or evenly-oriented;
- (b) $\alpha(H) = \alpha(T_H) + \sum_{C \in \mathcal{C}_H} \left\lfloor \frac{|V_C|}{2} \right\rfloor - d(H).$

Note that all cycles of G are also in H , Assertion (a) implies that each cycle (if any) of G^σ is odd or evenly-oriented. Hence (ii) holds.

Since x does not lie on any cycle of G , by Lemma 1.15(i) we have

$$d(G) = d(H). \quad (2.21)$$

Recall that y is also a pendant vertex of T_G adjacent to x and $T_H = T_G - x - y$, then by Lemma 2.3, Assertion (b) and (2.21) we have

$$\alpha(G) = \alpha(H) + 1 = \alpha(T_H) + \sum_{C \in \mathcal{C}_H} \left\lfloor \frac{|V_C|}{2} \right\rfloor - d(H) + 1 = \alpha(T_G) + \sum_{C \in \mathcal{C}_G} \left\lfloor \frac{|V_C|}{2} \right\rfloor - d(G).$$

Thus (iii) holds.

Case 2. G has a pendant cycle C_q . In this case, let x be the unique vertex of C_q of degree 3, $H^\sigma = G^\sigma - C_q^\sigma$ and $M^\sigma = H^\sigma + x$. It follows from Lemma 2.5(iii) that M^σ is lower-optimal. Applying the induction hypothesis to M^σ yields

- (c) each cycle of M^σ is odd or evenly-oriented;
- (d) $\alpha(M) = \alpha(T_M) + \sum_{C \in \mathcal{C}_M} \left\lfloor \frac{|V_C|}{2} \right\rfloor - d(M)$.

Assertion (c) and Lemma 2.5(i) imply that each cycle of G^σ is odd or evenly-oriented since $\mathcal{C}_G = \mathcal{C}_M \cup C_q$. Thus, (ii) holds.

Combining Lemma 2.5(ii), Lemma 2.5(iv) and Assertion (d) we have

$$\alpha(G) = \alpha(M) + \left\lfloor \frac{q}{2} \right\rfloor - 1 = \alpha(T_M) + \sum_{C \in \mathcal{C}_M} \left\lfloor \frac{|V_C|}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor - d(M) - 1. \quad (2.22)$$

As C_q is a pendant cycle of G , we have

$$d(G) = d(M) + 1. \quad (2.23)$$

Note that $T_M \cong T_G$ and $\left\lfloor \frac{q}{2} \right\rfloor + \sum_{C \in \mathcal{C}_M} \left\lfloor \frac{|V_C|}{2} \right\rfloor = \sum_{C \in \mathcal{C}_G} \left\lfloor \frac{|V_C|}{2} \right\rfloor$. Together with (2.22)-(2.23) we have

$$\alpha(G) = \alpha(T_G) + \sum_{C \in \mathcal{C}_G} \left\lfloor \frac{|V_C|}{2} \right\rfloor - d(G),$$

as desired. □

3 Proofs of main results

We will first provide the proof of Theorem 1.1, based on which the other proofs follow.

3.1 Theorem 1.1

Lemma 2.1 already established (1.1). We now characterize all the oriented graphs G^σ which attain the lower bound by considering the sufficient and necessary conditions for the equality in (1.1).

For “sufficiency”, we proceed by induction on the order n of G to show that G^σ is lower-optimal if G^σ satisfies the conditions (i)-(iii). The $n = 1$ case is trivial. Suppose that any oriented graph of order smaller than n which satisfies (i)-(iii) is lower-optimal, and suppose G^σ is an oriented graph with order $n \geq 2$ that satisfies (i)-(iii). Since the cycles (if any) of G^σ are pairwise vertex-disjoint, Lemma 1.15(iv) states that G has exactly $d(G)$ cycles, implying

that $|W_{\mathcal{C}}| = d(G)$. If T_G is an empty graph, it follows from (ii) that G^σ is an odd cycle or an evenly-oriented cycle, leading to the fact that G^σ is lower-optimal. So in what follows, we assume that T_G has at least one edge.

Note that $\alpha(T_G) = \alpha(\Gamma_G) + d(G) = \alpha(T_G - W_{\mathcal{C}}) + d(G)$. Then by Corollary 1.17(ii), there exists a pendant vertex of T_G not in $W_{\mathcal{C}}$. Thus G contains at least one pendant vertex, say y . Let x be the unique neighbor of y in G and let $H^\sigma = G^\sigma - x - y$. Then y is also a pendant vertex of T_G adjacent to x . By Lemma 2.3, we have

$$\alpha(T_G) = \alpha(T_G - x) = \alpha(T_H) + 1. \quad (3.1)$$

If $x \in W_{\mathcal{C}}$, then the graph $\Gamma_G \cup d(G)K_1$ can be obtained from $(T_G - x) \cup K_1$ by removing some edges. By Lemma 1.8(ii), we get

$$\alpha(\Gamma_G) + d(G) \geq \alpha(T_G - x) + 1. \quad (3.2)$$

Now from (3.1)-(3.2) we have $\alpha(\Gamma_G) \geq \alpha(T_G - x) - d(G) + 1 = \alpha(T_G) - d(G) + 1$, a contradiction to (iii).

Thus x does not lie on any cycle of G . Then y is also a pendant vertex of Γ_G adjacent to x and $\Gamma_H = \Gamma_G - x - y$. By Lemma 2.3 we have

$$\alpha(\Gamma_G) = \alpha(\Gamma_H) + 1. \quad (3.3)$$

As x does not lie on any cycle of G , Lemma 1.15(i) implies that

$$d(G) = d(H). \quad (3.4)$$

Now from condition (iii) and (3.1), (3.3)-(3.4), we have $\alpha(T_H) = \alpha(\Gamma_H) + d(H)$. Also note that all cycles of G are cycles of H , we conclude that H^σ satisfies conditions (i)-(iii). By induction hypothesis we have

$$sr(H^\sigma) + 2\alpha(H) = 2(n - 2) - 2d(H). \quad (3.5)$$

Furthermore, it follows from Lemmas 1.13 and 2.3 that

$$sr(G^\sigma) = sr(H^\sigma) + 2 \text{ and } \alpha(G) = \alpha(H) + 1. \quad (3.6)$$

By (3.4)-(3.6) we have $sr(G^\sigma) + 2\alpha(G) = 2n - 2d(G)$, implying that G^σ is lower-optimal.

For “necessity”, let G^σ be lower-optimal. By Lemma 2.7, the oriented cycles (if any) of G^σ are pairwise vertex-disjoint, and each oriented cycle of G^σ is odd or evenly-oriented. This implies (i) and (ii).

We proceed by induction on the order n of G to prove (iii). The $n = 1$ case is trivial. Suppose that (iii) holds for all lower-optimal oriented graph of order smaller than n , and suppose G^σ is a lower-optimal oriented graph of order $n \geq 2$.

If T_G is an empty graph, then G^σ is an odd cycle or an evenly-oriented cycle, in which case (iii) follows immediately. Now suppose T_G has at least one edge, then T_G has at least one pendant vertex, say y . Similar to before, either G contains y as a pendant vertex, or G contains a pendant cycle.

Case 1. G has a pendant vertex y .

Let x be the neighbor of y in G and $H^\sigma = G^\sigma - x - y$. By Lemma 2.6, x is not on any cycle of G and H^σ is also lower-optimal. Applying induction hypothesis to H^σ yields

$$\alpha(T_H) = \alpha(\Gamma_H) + d(H). \quad (3.7)$$

Since x does not lie on any cycle of G , Lemma 1.15(i) states that

$$d(G) = d(H). \quad (3.8)$$

Note that y is also a pendant vertex of T_G (resp. Γ_G) adjacent to x and $T_H = T_G - x - y$ (resp. $\Gamma_H = \Gamma_G - x - y$), then by Lemma 2.3 we have

$$\alpha(T_G) = \alpha(T_H) + 1 \text{ and } \alpha(\Gamma_G) = \alpha(\Gamma_H) + 1. \quad (3.9)$$

From (3.7)-(3.9) we have

$$\alpha(T_G) = \alpha(\Gamma_G) + d(G),$$

as desired.

Case 2. G has a pendant cycle C_q .

Let x be the unique vertex of C_q of degree 3 and $H^\sigma = G^\sigma - C_q^\sigma$. By Lemma 2.5(iii), H^σ is lower-optimal. Applying the induction hypothesis to H^σ yields

$$\alpha(T_H) = \alpha(\Gamma_H) + d(H). \quad (3.10)$$

From Lemma 2.5(ii) we have

$$\alpha(G) = \alpha(H) + \left\lfloor \frac{q}{2} \right\rfloor. \quad (3.11)$$

Note that $\mathcal{C}_G = \mathcal{C}_H \cup C_q$. Together with (3.11) and Lemma 2.7(iii) we have

$$\begin{aligned} \alpha(T_G) &= \alpha(H) + \left\lfloor \frac{q}{2} \right\rfloor - \sum_{C \in \mathcal{C}_G} \left\lfloor \frac{|V_C|}{2} \right\rfloor + d(G) \\ &= \alpha(H) - \sum_{C \in \mathcal{C}_H} \left\lfloor \frac{|V_C|}{2} \right\rfloor + d(G). \end{aligned} \quad (3.12)$$

Since H^σ is lower-optimal, Lemma 2.7(iii) states that

$$\alpha(T_H) = \alpha(H) - \sum_{C \in \mathcal{C}_H} \left\lfloor \frac{|V_C|}{2} \right\rfloor + d(H). \quad (3.13)$$

As C_q is a pendant cycle of G , we have

$$d(G) = d(H) + 1. \quad (3.14)$$

Combining (3.12)-(3.14) yields

$$\alpha(T_G) = \alpha(T_H) + 1. \quad (3.15)$$

Note that $\Gamma_G \cong \Gamma_H$, then the required equality $\alpha(T_G) = \alpha(\Gamma_G) + d(G)$ follows from (3.10) and (3.14)-(3.15). This completes the proof. \square

3.2 Theorems 1.4, 1.5 and 1.6

The proofs of Theorems 1.4, 1.5 and 1.6 follow almost directly from Theorem 1.1, and are rather similar to each other in nature. Here we only provide the proof of Theorem 1.4 and leave the rest to the readers.

The *join* of two disjoint graphs G_1 and G_2 , denoted by $G_1 \vee G_2$, is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2 by an edge. First we recall the following fact.

Lemma 3.1 ([11]). *Let G be an simple connected graph with n vertices and m edges. Then*

$$\frac{1}{2} \left[(2m + n + 1) - \sqrt{(2m + n + 1)^2 - 4n^2} \right] \leq \alpha(G) \leq \sqrt{n(n-1) - 2m} + \frac{1}{4} + \frac{1}{2}.$$

The equality on the right holds if and only if $G \cong K_{n-\alpha(G)} \vee \alpha(G)K_1$.

Proof of Theorem 1.4. Note that for a given simple connected graph G with $|V_G| = n$ and $|E_G| = m$, we have $d(G) = m - n + 1$. Together with (1.1) and Lemma 2.2, we have

$$sr(G^\sigma) + \alpha(G) \geq 4n - 2m - 2 - \alpha(G) \geq 4n - 2m - \sqrt{n(n-1) - 2m} + \frac{1}{4} - \frac{5}{2}$$

as stated in (1.2).

Now we prove the sufficient and necessary conditions for equality in (1.2).

“Sufficiency:” First consider the case that $G \cong S_n$. If $n = 1$, then (1.2) holds trivially. If $n \geq 2$, then we have $sr(G^\sigma) = 2$ and $\alpha(G) = n - 1$. Together with the fact that $m = n - 1$ we have that equality holds in (1.2).

Now we consider the case $G \cong C_3$. By Lemma 1.12 we have $sr(G^\sigma) = 2$. Note that in this case $\alpha(G) = 1$ and $m = n = 3$. Hence we have equality in (1.2).

“Necessity:” Combining Theorem 1.1 and Lemma 2.2 we have that the equality in (1.2) holds if and only if G^σ is lower-optimal and $G \cong K_{n-\alpha(G)} \vee \alpha(G)K_1$. Note that the cycles (if any) of G^σ are pairwise vertex-disjoint. Hence, $n - \alpha(G) = 1$ or $n - \alpha(G) = 2$ and $\alpha(G) = 1$, which implies $G \cong S_n$ or $G \cong C_3$.

This completes the proof. \square

4 Concluding remarks

It is well-known that the AutoGraphiX system determines classes of extremal or near-extremal graphs with a variable neighborhood search heuristic. As part of a larger study [4], the AutographiX2 (AGX2) [5, 7, 8] system was used to study the following type of problems. For each pair of graph invariants $i_1(G)$ and $i_2(G)$, eight bounds of the following form were considered:

$$\underline{b} \leq A \cdot i_1(G) \oplus B \cdot i_2(G) \leq \bar{b}, \quad (4.1)$$

where \oplus denotes one of the operations $+$, $-$, \times , $/$, A, B are two constants, while \underline{b} and \bar{b} are, respectively, lower and upper bounding functions. In this paper we considered the invariants

$i_1(G) = sr(G^\sigma)$ and $i_2(G) = \alpha(G)$ where G is the underlying graph of G^σ . Theorem 1.1 provides sharp lower bound on $sr(G^\sigma) + 2\alpha(G)$; whereas Theorems 1.4, 1.5 and 1.6 provide sharp lower bounds on $sr(G^\sigma) + \alpha(G)$, $sr(G^\sigma) - \alpha(G)$ and $sr(G^\sigma)/\alpha(G)$.

It is nature to extend this study through examining the following bounds:

- sharp upper bounds on $sr(G^\sigma) + 2\alpha(G)$;
- sharp upper bounds on $sr(G^\sigma) + \alpha(G)$, $sr(G^\sigma) - \alpha(G)$ and $sr(G^\sigma)/\alpha(G)$;
- sharp upper and lower bounds on $sr(G^\sigma) \cdot \alpha(G)$.

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